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ON THE UNIQUENESS OF ANGULAR FREQUENCY USING HARMONIC BALANCE FROM THE EQUATION OF MOTION AND THE ENERGY RELATION

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Interesting discussions have been made recently in references [1–5] on the uniqueness of angular frequency using harmonic balance from the equation of motion,

$$\ddot{x} + x^3 = 0, \qquad x(0) = x_0, \qquad \dot{x}(0) = 0, \tag{1}$$

and the energy relation,

$$(\dot{x})^2 = \tfrac{1}{2}(x_0^4 - x^4). \tag{2}$$

Here overdots denote differentiation with repect to time, τ. It is noted that the inclusion of higher order harmonics in the method of harmonic balance gives better argreement between the values of the angular frequency as determined from equations (1) and (2). The non-linear differential equation (1) is a good test equation for which the exact solution exists in the form of an elliptic function. In order to investigate further, a better non-linear equation,

$$\ddot{x} + x^3/(1 + \lambda x^2) = 0, \qquad \lambda > 0, \tag{3}$$

as suggested by Mickens [5] is considered here.

Multiplying equation (3) by $2\dot{x}$ and using the initial conditions

$$x = x_0, \quad \dot{x} = 0 \qquad \text{at } \tau = 0, \tag{4}$$

after integration one obtains the energy relation

$$(\dot{x})^2 = I(x_0) - I(x), \tag{5}$$

where $I(x) = \int_0^x \{2x^3/(1 + \lambda x^2)\}\,\mathrm{d}x = x^2/\lambda - \ln(1 + \lambda x^2)/\lambda^2$.

The restoring force function in the equation of motion (3) is an odd function. The behaviour of oscillations is the same for both negative and positive amplitudes.

From equations (4) and (5), one obtains

$$\int_{x=0}^{x=x_0} \frac{\mathrm{d}x}{\sqrt{I(x_0) - I(x)}} = \int_{\tau=0}^{\tau=T/4} \mathrm{d}\tau = \frac{T}{4} = \frac{\pi}{2\omega}, \tag{6}$$

where T is the period and ω is the angular frequency. It should be noted that the integrand in equation (6) has a pole at the end point of the integration (i.e., at $x = x_0$) which may affect the accuracy of an integration rule adversely. In such a situation, the general procedure, as suggested in reference [6], is to modify the integrand by subtracting from it an expression (integrable in closed form) which eliminates the singularity and yields a

0022–460X/97/080367 + 04 \$25.00/0/sv950679

form which can be integrated numerically. For the present case, equation (6) is written in the form

$$\frac{\pi}{2\omega} = \frac{\pi}{2} f(x_0) + \int_0^{x_0} \left\{ \frac{1}{\sqrt{I(x_0) - I(x)}} - \frac{f(x_0)}{\sqrt{x_0^2 - x^2}} \right\} \mathrm{d}x, \tag{7}$$

where $f(x_0) = \sqrt{(1 + \lambda x_0^2)}/x_0$. A ten-point Gauss rule was adopted here for evaluating the integral in equation (7).

Representing the restoring force function as a polynomial, one can write the equation of motion (3) in the form

$$\ddot{x} + x^3 \sum_{m=0}^{N} a_m (\lambda x^2)^m = 0, \tag{8}$$

where $a_0 = 1$ and the constants a_m are obtained through the least-squares curve fit of the function $1/(1 + \lambda x^2)$ for the specified range of x.

The energy relation becomes

$$(\dot{x})^2 = \sum_{m=0}^{N} \frac{a_m \lambda^m}{(m+2)} \left\{ (x_0^2)^{m+2} - (x^2)^{m+2} \right\}. \tag{9}$$

For the lowest order harmonic, the periodic solution which satisfies the initial conditions (4) is

$$x = x_0 \cos(\omega\tau) \tag{10}$$

Substituting equation (10) in equations (8) and (9), and neglecting the higher order harmonic, one obtains angular freqencies ω_{EM} and ω_{ER}, corresponding to the equation of motion (8) and the energy relation (9), as

$$\omega_{EM}^2 = x_0^2 \left(\tfrac{3}{4} + \tfrac{1}{4} \sum_{m=1}^{N} a_m \left\{ \frac{\lambda x_0^2}{4} \right\}^m C(2m+3, m+1) \right), \tag{11}$$

$$\omega_{ER}^2 = x_0^2 \left(\tfrac{5}{8} + 2 \sum_{m=1}^{N} \frac{a_m (\lambda x_0^2)^m}{(m+2)} \left\{ 1 - C(2m+4, m+2)/4^{m+2} \right\} \right), \tag{12}$$

where the binomial coefficient, $C(n, r) = n!/(r!(n-r)!)$.

Since the behaviour of oscillations is the same for both negative and positive amplitudes, and $\dot{x}$ becomes zero when x is $-x_0$ or $+x_0$, the right side of the energy relation (9) has $(x_0^2 - x^2)$ as a common factor. This is the reason why the integrand in equation (6) has a pole at $x = x_0$ when the energy relation is integrated from $\tau = 0$ to the quarter period, $\tau = T/4$. The energy relation (9) can be written in the form

$$(\dot{x})^2/(x_0^2 - x^2) = \sum_{m=0}^{N} \frac{a_m \lambda^m}{(m+2)} \left\{ (x_0^2)^{m+1} + \sum_{n=1}^{(m+1)} (x_0^2)^{m+1-n} (x^2)^n \right\}. \tag{13}$$

Substituting equation (10) in equation (13), and neglecting the higher order harmonic, one obtains the angular freqency ω_{ER} as

$$\omega_{ER}^2 = x_0^2\left(\frac{3}{4} + \sum_{m=1}^{N}\frac{a_m(\lambda x_0^2)^m}{(m+2)}\left\{1 + \sum_{n=1}^{(m+1)} C(2n, n)/4^n\right\}\right), \tag{14}$$

In order to verify the accuracy of the results, the constants a_m in the polynomial which represents the function $1/(1+\lambda x^2)$ are obtained through the least-squares curve fit by considering $N = 3$, for the range of $|\sqrt{\lambda}x| \leqslant 1$. These are $a_1 = -0{\cdot}97040$, $a_2 = 0{\cdot}742183$ and $a_3 = -0{\cdot}27618$. Equations (11), (12) and (14) are written in a simplified form as

$$\omega_{EM}^2 = x_0^2\left(\tfrac{3}{4} + \tfrac{5}{8}a_1(\lambda x_0^2) + \tfrac{35}{64}a_2(\lambda x_0^2)^2 + \tfrac{63}{128}a_3(\lambda x_0^2)^3\right), \tag{15a}$$

$$\omega_{ER}^2 = x_0^2\left(\tfrac{5}{8} + \tfrac{11}{24}a_1(\lambda x_0^2) + \tfrac{93}{256}a_2(\lambda x_0^2)^2 + \tfrac{193}{640}a_3(\lambda x_0^2)^3\right), \tag{15b}$$

$$\omega_{ER}^2 = x_0^2\left(\tfrac{3}{4} + \tfrac{5}{8}a_1(\lambda x_0^2) + \tfrac{35}{64}a_2(\lambda x_0^2)^2 + \tfrac{63}{128}a_3(\lambda x_0^2)^3\right), \tag{15c}$$

Since the expressions in equations (15a) and (15c) are identical, the values of angular frequency obtained from the equation of motion and the energy relation are the same for the lowest order harmonic. As one should expect, with the energy relation being the first integral of the equation of motion, the two procedures have given exactly the same solution.

The numerically integrated values of ω given in Table 1 for a negligibly small value of λ (say, 0·0001) are found to be in good agreement with the exact solution for $\lambda = 0$ [4]. Two factors motivated the present exercise: one is whether the lowest order harmonic solution is the cause of the disagreement in the values of ω as obtained from the equation of motion and the energy relation, or whether the reason lies elsewhere. It is demonstrated here that the discrepency is due to the singularity in $\dot{x}$ (at $\tau = 0$) creeping into the energy relation.

TABLE 1

Comparison of angular frequencies

		Exact integration,	Harmonic balance method		
λ	x_0	ω, equation (6)	ω_{EM}, equation (15a)	ω_{ER}, equation (15b)	ω_{ER}, equation (15c)
0·0001	0·25	0·21172	0·21651	0·19764	0·21651
	0·50	0·42359	0·43301	0·39528	0·43301
	1·00	0·84717	0·86599	0·79054	0·86599
1·000	0·25	0·20677	0·21119	0·19336	0·21119
	0·50	0·38736	0·39421	0·36379	0·39421
	1·00	0·63678	0·64300	0·60545	0·64300

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